Comment. Math.
Univ. St. Pauli
XXVI—2, 1977

Reduced Powers and Models for Infinitely Logic

by

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(Received September 28, 1977)

Introductin.

In (1), Mansfield defines the Boolean ultrapower which is a generalization of the ultrapower operation. The Boolean ultrapower operation consists of two sub-operations, i.e. one is the method of constructing a $\boldsymbol{B}$-valued structure $\mathfrak{A}^{(B)}$ from a two-valued structure $\mathfrak{A}$ and the other is the method of constructing a two-valued structure $\mathfrak{A}^{(B)}/\sigma$ from the $\boldsymbol{B}$-valued structure $\mathfrak{A}^{(B)}$ by an ultrafilter σ on $\boldsymbol{B}$ (where $\boldsymbol{B}$ is a complete Boolean algebra).

In §1 of this paper, we shall define the "reduced power operation" which is to be a method of constructing a Boolean-valued structure from another Boolean-valued structure. The two sub-operations mentioned above are special cases of the new operation we define here. In §2 and §3, we shall show that the fundamental properties of Boolean ultrapowers hold for reduced powers. Section 4 deals with a semantical characterization of $\boldsymbol{L}_{\omega_1,\omega}$ using this result. Then, "The Model Existence Theorem of $\boldsymbol{L}_{\omega_1,\omega}$" can be proved easily using this characterization.

We shall use the usual set theoretical notations. The letters α, β, γ, κ will be used to denote cardinals, and λ, ν, ξ, η to denote ordinals, ω is to be the set of natural numbers, ω_1 the first uncountable cardinal, and k, 1, $\cdots$ will denote natural numbers. Letting α^β denote cardinal exponentiation, we can write $\alpha^{(\beta)}=\mathrm{Sup}\,\{\alpha^\gamma;\ \gamma<\beta\}$. We let ${}^J I$ be the set of all function from J to I, where Card(I) is the cardinality of I. If i varies over I, then $\vec{i}$ varies over the elements of ${}^\eta I$ for some η and is identified with a η-sequence $\langle i(\xi);\ \xi<\eta\rangle$ of I. The letters $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ denote complete Boolean algebras which will be identified with their universes. Finally, we use the symbols $\cdot$, $+$, $-$, $\sum$, $\prod$, $<$, $\leqq$, $\mathbf{0}$, $\mathbf{1}$ to stand for Boolean operations, notations, and constants.

§.1. Boolean structurs which are induced by homomorphisms.

Let $\boldsymbol{L}$ be a first order language with equality and, by $\boldsymbol{L}(\mathrm{A})$, we shall denote the first order language which is obtained from $\boldsymbol{L}$ by adding the set of new constant symbols $\{\underline{a};\ a\in A\}$. We shall fix the first order language $\boldsymbol{L}$ thoughtout this paper. Also, we shall use the infinite language $\boldsymbol{L}(A)_{\alpha,\beta}$ for each infinite cardinal α and β, which is obtained from $L(A)$ by adding conjunctions and disjunctions of less than α formulas and quantifications over fewer than β individual variables. A formula in $L(A)$ means a formula in $L(A)_{\alpha,\beta}$ for some α, β. A sentence (a closed term) means a formula (a term) which contains no free variables.

Let $\boldsymbol{M}$ be a B-valued structure (for $\boldsymbol{L}$), i.e. $\boldsymbol{M}$ is a mapping whose domain is the union of $\{\phi\}$ and the set of nonlogical symbols in $\boldsymbol{L}$ such that:

$\boldsymbol{M}(\phi)$ is a nonempty set, denoted M,

$\boldsymbol{M}(P)$ is a function from ${}^k M$ to $\boldsymbol{B}$, for any k-ary predicate symbol P in $\boldsymbol{L}$,

$\boldsymbol{M}(f)$ is a function from ${}^k M$ to M, for any k-ary function symbol f in $\boldsymbol{L}$, and

$\boldsymbol{M}(e)$ is in M, for any constant symbol e in $\boldsymbol{L}$.

For any closed term t any sentence ψ in $\boldsymbol{L}(M)$, the value of t in M and the value of ψ in $\boldsymbol{B}$ are defined as usual and denoted by $[t](\boldsymbol{M})$ and $|\psi|(\boldsymbol{M})$ respectively. Where no confusion is likely to occur, we shall omit $(\boldsymbol{M})$ in $[t](\boldsymbol{M})$ and $|\psi|(\boldsymbol{M})$ and identify $\underline{m}$ with m for any m in M.

We say that $\boldsymbol{M}$ satisfies the equality axioms if $|\psi|=1$ for any equality axiom ψ in $\boldsymbol{L}$.

We say that $\boldsymbol{M}$ is rigid if $|\mathrm{m}=m'|<1$ for any $m, m'(\neq)$ in M.

Suppose σ is a homomorphism from $\boldsymbol{B}$ to $\boldsymbol{C}$ and $\boldsymbol{M}$ is a $\boldsymbol{B}$-valued structure satisfying the equality axioms. We shall define the $\boldsymbol{C}$-valued structure $\boldsymbol{X}$ as follows:

$X(=\boldsymbol{X}(\phi))$ is the set of all functions x from M to $\boldsymbol{C}$ satisfying

1) $\Sigma\{x(m);\ m\in M\}=1$,

2) $x(m)\cdot x(m')=\sigma(|m=m'|)$ for all m, m' in M, and $\boldsymbol{X}(P), \cdots, \boldsymbol{X}(f), \cdots, \boldsymbol{X}(e), \cdots$ are defined by

$\boldsymbol{X}(P)(x_1, \cdots, x_k)=\sum\{\sigma(\boldsymbol{M}(P)(m_1, \cdots, m_k))\cdot x_1(m_1)\cdots x_k(m_k);\ m_1, \cdots, m_k\in M\}$ for any P in $\boldsymbol{L}$,

$\boldsymbol{X}(f)(x_1, \cdots, x_k)(m)=\sum\{\sigma(|m=fm_1\cdots m_k|)\cdot x_1(m_1)\cdots x_k(m_k);\ m_1, \cdots, m_k\in M\}$ for any f in $\boldsymbol{L}$, and $\boldsymbol{X}(e)(m)=\sigma(|m=e|)$ for any e in $\boldsymbol{L}$.

We can easy check that the definitions of $\boldsymbol{X}(f)$ and $\boldsymbol{X}(e)$ are well defined. $\boldsymbol{X}$ is called the reduced power of $\boldsymbol{M}$ and σ, denoted by $\boldsymbol{M}^{(\sigma)}$.

Remark 1. In (1), $\boldsymbol{N}^{(\boldsymbol{B})}$ is defined for any two-valued structure $\boldsymbol{N}$. Let σ be a homomorphism from $\langle\mathbf{0}, \mathbf{1}\rangle$ to $\boldsymbol{B}$. Then, $\boldsymbol{N}^{(\boldsymbol{B})}$ is isomorphic to $\boldsymbol{N}^{(\sigma)}$. Also, in (1), $\boldsymbol{M}/\mu$ is defined for any homomorphism μ: $\boldsymbol{B}\to\langle\mathbf{0}, \mathbf{1}\rangle$ and for any $\boldsymbol{B}$-valued structure $\boldsymbol{M}$. Then, $\boldsymbol{M}/\mu$ is isomorphic to $\boldsymbol{M}^{(\mu)}$. So, a Boolean ultrapower $\boldsymbol{N}^{(\boldsymbol{B})}/\mu$ (see (1)) is isomorphic to $(\boldsymbol{N}^{(\sigma)})^{(\mu)}$.

Henceforth, we let $\sigma: \boldsymbol{B}\to\boldsymbol{C}$ be a homomorphism, $\boldsymbol{M}$ a $\boldsymbol{B}$-valued structure satifying the exuality axioms and $\boldsymbol{X}=\boldsymbol{M}^{(\sigma)}$. We also let

$m, m', \cdots$ vary in $M, x, x', \cdots$ vary in $X, b, b', \cdots$ vary in $\boldsymbol{B}$ and $c, c', \cdots$ vary in $\boldsymbol{C}$. The symbols $\psi, \theta, \cdots$ will be used to express formulas and $\psi(\vec{v})$ to stand for a formula ψ such that every free variables in ψ occurs in a sequence in $\vec{v}$.

LEMMA 1. *X satisfies the equality axioms.*

Proof. This is easy to check using the definitions (1), (2) of X and $|x=x'|=\sum_m x(m)\cdot x'(m)$.

We define the mapping $*$ from M to X by $m^*(m')=\sigma(|m=m'|)$ for all m' in M. It is easy to check that m^* satifies the definitions of X, i.e. m^* is in X.

In particular, $|m^*=x|=x(m)$ and $|m^*=m'^*|=\sigma(|m=m'|)$. Hence we have the following.

LEMMA 2. *X is rigid.*

§. 2. Relations between $\boldsymbol{M}$ and $\boldsymbol{M}^{(\sigma)}$.

It is well known that every ultrapower of $\boldsymbol{N}$ is an elementary extension of $\boldsymbol{N}$ for any two-valued structure $\boldsymbol{N}$. We shall define an elementary extension in the sense of Boolean and show that "$\boldsymbol{X}$ is an Boolean elementary extension of $\boldsymbol{M}$" also holds under the conditions of σ and $\boldsymbol{M}$ stated explicitly below.

Let $\boldsymbol{X}'$ be a $\boldsymbol{C}$-valued structure, u be a mapping from M to $X'(=\boldsymbol{X}'(\phi))$ and ψ be a set of formulas in $\boldsymbol{L}$. u is called a $(\boldsymbol{\Gamma}, \sigma)$-elementary mapping, if $\sigma(|\psi(\vec{m})|)=|\psi(u(\vec{m}))|$ holds for any $\psi(\vec{v})$ in $\boldsymbol{\Gamma}$ and for any sequence $\vec{m}$ of M where $u(\vec{m})$ denotes $\langle u(m_\eta);\ \eta<lh(\vec{m})\rangle$ and $\vec{m}=\langle m_\eta;\ \eta<lh(\vec{m})\rangle$. In particular, $\boldsymbol{B}=\boldsymbol{C}$ and σ is the identity, and u is called a $\boldsymbol{\Gamma}$-elementary mapping. A homomorphism σ is said to be κ-complete, if $\sigma(\sum\{b_\xi;\ \xi<\eta\})=\sum\{\sigma(b_\xi);\ \xi<\eta\}$ holds for any $\eta<\kappa$ and for any subset $\{b_\xi;\ \xi<\eta\}$ of $\boldsymbol{B}$, and σ is complete, if σ is κ-complete for any κ.

THEOREM 1. *Suppose σ is complete. For any formule $\psi(v_1, \cdots, v_k)$ in $L_{\alpha,\omega}$ and for arbitrary elements $x_1, \cdots, x_k$ in X, the equality $|\psi(x_1, \cdots, x_k)|=\sum\{\sigma(|\psi(m_1, \cdots, m_k)|)\cdot x_1(m_1)\cdot \cdots x_k(m_k);\ m_1, \cdots, m_k\in M\}$ holds.*

Proof. Use induction on the construction of formulas.

Case 1. ψ is atomic. Then it is clear from the definition.

Case 2. ψ is $\neg\theta$. By the induction hypothesis, we have

$$\begin{aligned}|\psi(x_1, \cdots, x_k)| &= -|\theta(x_1, \cdots, x_k)| \\ &= -\sum_{m_1,\ldots,m_k}(\sigma(|\theta(m_1, \cdots, m_k)|)\cdot x_1(m_1)\cdots x_k(m_k))\end{aligned}$$

Put

$$\square m_1, \cdots, m_k \square = \sigma(|\theta(m_1, \cdots, m_k)|) \cdot x_1(m_1) \cdots x_k(m_k) ,$$
$$[m_1, \cdots, m_k] = \sigma(|\psi(m_1, \cdots, m_k)|) \cdot x_1(m_1) \cdots x_k(m_k) .$$

We want to prove $-\sum_{m_1,\ldots,m_k} \square m_1, \cdots, m_k \square = \sum_{m_1,\ldots,m_k} [m_1, \cdots, m_k]$. It will suffice to show (1) and (2),

(1) For any $c>0$ in C, there exists $m_1, \cdots, m_k$ in M such that $\square m_1, \cdots, m_k \square \cdot c>0$ or $[m_1, \cdots, m_k] \cdot c>0$ holds.

(2) For any $m_1, \cdots, m_k, m_1', \cdots, m_k'$ in M, $\square m_1, \cdots, m_k \square \cdot [m_1, \cdots, m_k]=0$.

To show (1), Let $c>0$ in C. Using $\sum_{m_1,\ldots,m_k} x_1(m_1) \cdots x_k(m_k)=1$, we want to choose $m_1, \cdots, m_k$ in M such that $c'=c \cdot x_1(m_1) \cdots x_k(m_k)>0$.

To show (2). This is a simple calculation using the fact that σ is complete, so we omit it here.

Case 3. ψ is $\bigvee_{\eta<\xi} \theta_\eta$ where $\xi<\alpha$. By the induction hypothesis, we have

$$\begin{aligned} |\psi(x_1, \cdots, x_k)| &= \textstyle\sum_{\eta<\xi} |\theta_\eta(x_1, \cdots, x_k)| \\ &= \sum_{\eta<\xi} \sum_{m_1,\ldots,m_k} (\sigma(|\theta_\eta(m_1, \cdots, m_k)|) \cdot x_1(m_1) \cdots x_k(m_k)) \end{aligned}$$

Since σ is complete,

$$\begin{aligned} &|\psi(x_1, \cdots, x_k)| \\ &= \sum_{m_1,\cdots,m_k} \left(\sigma\left(\sum_{\eta<\xi} |\theta_\eta(m_1, \cdots m_k)|\right) \cdot x_1(m_1) \cdots \cdot x_k(m_k)\right) \\ &= \sum_{m_1,\cdots,m_k} \sigma\left(\left|\bigvee_{\eta<\xi} \theta_\eta(m_1, \cdots, m_k)\right|\right) \cdot x_1(m_1) \cdots \cdot x_k(m_k) . \end{aligned}$$

Case 4. ψ is $(\exists v)\theta$. This is same as case 3.

COROLLARY 1. *If σ is comlete, then $*$ is a $(L_{\alpha,\omega}, \sigma)$-elementary mapping.*

Proof. In Theorem 1, substitute $m_1^*, \cdots, m_k^*$ for $x_1, \cdots, x_k$ respectively. It is easy to see that $|\psi(m_1^*, \cdots, m_k^*)| = \sigma(|\psi(m_1, \cdots, m_k)|)$ hold for any formula $\psi(v_1, \cdots, v_k)$ in $L_{\alpha,\omega}$.

C satisfies the (I, J)-distributive law, if $\sum_{h \in J^I} \prod_{i \in I} c_{i,h(i)} = \prod_{i \in I} \sum_{j \in J} c_{i,j}$ holds for any subset $\{c_{i,j};\ i \in I,\ j \ni J\}$ of C.

The next Theorem is an extension of Therem 1.

THEOREM 2. *Let $\alpha \geqq \beta \geqq \omega$. Let γ be a cardinal of M. Assume σ is α-complete and and $\gamma^{(\beta)}$-complete, and that C satisfies the (κ', γ)-distributive law for any $\kappa'<\beta$. Then, $|\psi(\vec{v})| = \sum_{\vec{m}} (\sigma(|\psi(\vec{m})|) \cdot \prod_{\eta<\lambda} x_\eta(m_\eta))$ holds for any formula $\psi(\vec{v})$ in $L_{\alpha,\beta}$ (where $1h(\vec{v})=\lambda<\beta$) and for any sequence $\vec{x}=\langle x_\eta;\ \eta<\lambda\rangle$ of X.*

Theorem 2 will be proved by the same method as in the proof of Theorem 1.

COROLLARY 2. *Under the same assumption of Therem 2, $*$ is a*

$(L_{\alpha,\beta}, \sigma)$*-elementary mapping.*

Suppose M satisfies the condition that for any formula $\psi(v_0)$ in $L(M)_{\omega,\omega}$, there exists a exists a m in M such that $|(\exists v_0)\psi(v_0)|=|\psi(m)|$. Then, we can prove without any assumptions on σ that $*$ is a $(L_{\omega,\omega}, \sigma)$-elementary mapping.

§. 3. Some properties of induced powers.

Let Γ be a set of formulas. Then M is said to have Γ-witness if and only if for any formula $\psi(\vec{v})$ in Γ there exists some $\vec{x}$ in ${}^{\lambda}M$ (where $\lambda=lh(\vec{v})$) such that $|(\exists\vec{v})\psi(\vec{v})|=|\psi(\vec{x})|$. M is said to have a witness iff M has a Γ-witness for any set Γ.

We shall show that X has a witness by using the following lemmas.

LEMMA 3. *Let* $\{m_i; i\in I\}$ *be a subset of* M *and* $\{c_i; i\in I\}$ *a subset of* C. *If* $\sigma(|m_i=m_j|)\geqq c_i\cdot c_j$ *for any* $i, j\in I$, *then there exists* x_0 *in* X *such that* $|x_0=m_i^*|\geqq c_i$ *for any* $i\in I$.

Proof. Put $c_0=-\sum_{i\in I} c_i$ and let m_0 be an arbitrary fix element of M. Define x_0: $M\to C$ by

$$x_0(m)=c_0\cdot\sigma(|m=m_0|)+\sum_{i\in I}(\sigma(|m=m_i|)\cdot c_i)$$

for any m in M. From the definition of X, if follows that x_0 belongs X. Also, it is easy to verify that $|x_0=m_i^*|\geqq c_i$. This mean that x_0 has the desired properties.

LEMMA 4. *Let* $\{x_i; i\in I\}$ *be a subset of* X *and* $\{c_i; i\in I\}$ *a subset of* C. *If* $|x_i=x_j|\geqq c_i\cdot c_j$ *for any* $i, j\in I$, *then there exist* x_0 *in* X *such that* $|x_0=x_i|\geqq c_i$ *for any* $i\in I$.

Proof. We defihe $h(m)$ in C for any m in M *by*

$$h(m)=\sum_{i\in I} x_i(m)\cdot c_i\ .$$

Then we have $h(m)\cdot h(m')\leqq\sigma(|m=m'|)$ for any m, m' in M. Therefore $\{h(m); m\in M\}$ and $\{m; m\in M\}$ satisfy the assumptions of Lemma 3. Choose x_0 in X such that $|x_0=m^*|\geqq h(m)$ for any m in M. This x_0 has the desire properties.

By using Lemma 4, we obtain the following Lemma 5.

LEMMA 5. *Let* $\{\vec{x}_i; i\in I\}$ *be a subset of* ${}^{\lambda}X$ *(where* $\vec{x}_i=\langle x_{i,\eta}; \eta<\lambda\rangle$ *for any* $i\in I$) *and* $\{c_i; i\in I\}$ *a subset of* C. *If* $|x_{i,\eta}=x_{j,\eta}|\geqq c_i\cdot c_j$ *for any* $i, j\in I$ *and any* $\eta>\lambda$, *then there exists* $\vec{x}=\langle x_\eta; \eta<\lambda\rangle$ *such that* $|x_\eta=x_{i,\eta}|\geqq c_i$ *for any* $i\in I$ *and any* $\eta>\lambda$.

THEOREM 3. *X has a witness.*

Proof. Let $\psi(\vec{v})$ be a formula in $L(M)$, $\lambda=lh(\vec{v})$, $\kappa=\mathrm{Card}\,({}^{\lambda}X)$ and $\langle\vec{x}_\eta; \eta<\lambda\rangle$ an enumeration of ${}^{\lambda}X$, where $\vec{x}=\langle x_{\eta,\nu}; \nu<\lambda\rangle$ for any $\eta<\lambda$.

Then, we have $|(\exists\vec{v})\psi(\vec{v})|=\sum_{\eta<k}|\psi(\vec{x}_\eta)|$.

Put
$$c_\eta=|\psi(\vec{x}_\eta)|,$$
$$\hat{c}_\eta=c_\eta\cdot-\sum_{\xi<\eta}c_\xi \quad \text{for any} \quad \eta<\kappa\ .$$

Then,
$$\sum_{\eta<\kappa}c=\sum_{\eta<\kappa}\hat{c}=|(\exists\vec{v})\psi(\vec{v})| \quad \text{and}$$
$$\hat{c}_\xi\cdot\hat{c}_\eta=\mathbf{0} \quad \text{for any} \quad \xi,\eta<\kappa(\xi\neq\eta)\ .$$

Using Lemma 5, choose $\vec{y}=\langle y_\nu;\nu<\lambda\rangle$ in ${}^\lambda X$ such that $|y_\nu=x_{\eta,\nu}|\geqq\hat{c}_\eta$ for any $\eta<\kappa$ and $\nu<\lambda$. Since X satisfies the equality axioms, $|\psi(\vec{y})|\geqq\hat{c}_\eta$ holds for any $\eta<\lambda$. Thus, $|\psi(\vec{y})|\geqq|(\exists\vec{v})\psi(\vec{v})|$. On the other hand, the inequality $|\psi(\vec{y})|\leqq|(\exists\vec{v})\psi(\vec{v})|$ is obvious. This $\vec{y}$ is the desired one.

COROLLARY 3. *Let N be a $\boldsymbol{B}$-valued structure. Then, there exists a rigid witness $\boldsymbol{B}$-valued structure $\boldsymbol{N}'$ such that $\boldsymbol{N}(\phi)\subseteqq\boldsymbol{N}'(\phi)$ and $|\psi(\vec{n})|(\boldsymbol{N})=|\psi(\vec{n})|(\boldsymbol{N}')$ for any $\psi(\vec{v})$ in $\boldsymbol{L}_{\alpha,\omega}$ and any $\vec{n}$ in ${}^\lambda N$ (where $N=\boldsymbol{N}(\phi)$).*

Proof. This Corollary is clear from Theorem 3 and Corollary 1.

Remark 2. In the construction of $\boldsymbol{M}^{(\sigma)}$, we could have used another definition. i.e. We shall define a $\boldsymbol{C}$-valued structure $\boldsymbol{Z}$ whose universe $Z(=\boldsymbol{Z}(\phi))$ is the set of all functions z from M to C satisfying

1)′ $\sum_m z(m)=\mathbf{1}$,

2)′ $z(m)\cdot z(m')=\mathbf{0}$ for any $m, m'(\neq)$ in M, and $\boldsymbol{Z}(P),\cdots,\boldsymbol{Z}(f),\cdots,$ $\boldsymbol{Z}(e),\cdots$ are defined by

$$\boldsymbol{Z}(P)(z_1,\cdots,z_k)=\sum_{m_1,\cdots,m_k}(\sigma(\boldsymbol{M}(P)(m_1,\cdots,m_k))\cdot z_1(m_1)\cdot\cdots\cdot z_k(m_k))\ ,$$
$$\boldsymbol{Z}(f)(z_1,\cdots,z_k)(m)=\sum_{m=fm_1\cdots m_k}z_1(m_1)\cdot\cdots\cdot z_k(m_k)$$

for any m in M, and,

$$\boldsymbol{Z}(e)(m)=\begin{cases}\mathbf{1} & (\text{if} \quad m=[e](\boldsymbol{M}))\ ,\\ \mathbf{0} & (\text{otherwise})\ .\end{cases}$$

In general $\boldsymbol{Z}$ and $\boldsymbol{X}$ are distinct. So, let $\tilde{z}$ be a function from M to $\boldsymbol{C}$ defined by $\tilde{z}(m)=\sum_{m'}z(m')\cdot\sigma(|m=m'|)$ for and any m in M, where z is in Z. It is easy to see that $\tilde{z}$ belongs to X for any z in Z. So, $\sim$ is a mapping from Z to X. It is also easy to verify that $\sim$ is onto and a $\boldsymbol{L}_{\alpha,\beta}$-elementary mapping for any α,β. Therefore, $\boldsymbol{X}$ and $\boldsymbol{Z}$ are isomorphic in the sence of $\boldsymbol{C}$-valued structures. But generally speaking $\boldsymbol{Z}$ is not rigid and it is more complex to show that $\boldsymbol{Z}$ has a witness. So, we shall employ $\boldsymbol{X}$ as the definition of reduced power.

Remark 3. Let σ^1 be a homomorphism from $\boldsymbol{B}$ to $\boldsymbol{C}$, σ^2 a homomorphism from $\boldsymbol{C}$ to $\boldsymbol{D}$ and $\boldsymbol{M}$ be a $\boldsymbol{B}$-valued structure. Let $\tau=\sigma^2\cdot\sigma^1$.

Then, it is possible to construct a $\boldsymbol{D}$-valued structure $\boldsymbol{M}^{(\tau)}$, denoted by $\boldsymbol{N}^0$, and $(\boldsymbol{M}^{(\sigma^1)})^{(\sigma^2)}$, denoted by $\boldsymbol{N}^1$.

Let n be in $N^0(=\boldsymbol{N}^0(\phi))$. We shall define a mapping n^* as follows:

$$n^*(g)=\sum_m \sigma^2(g(m))\cdot n(m) \text{ for any } g \text{ in } G(=\boldsymbol{M}^{(\sigma^1)}(\phi)) .$$

It is easy to check that n^* is in $N^1(=\boldsymbol{N}^1(\phi))$. So, $*$ is a mapping from N^0 to N^1. And, $\boldsymbol{N}^0(P)(n_1, \cdots, n_k)=\boldsymbol{N}^1(P)(n_1^*, \cdots, n_k^*)$ hold for any $n_1, \cdots, n_k$ in N^0. $*$ is a Qf-elementary mapping where Qf is the set of atomic formulas in $\boldsymbol{L}$. But generally, $*$ is neither onto nor a $\boldsymbol{L}_{\alpha,\beta}$-elementary mapping. For $*$ to be a $\boldsymbol{L}_{\alpha,\beta}$-elementary mapping, futher conditions are necessary for σ^1, σ^2 and $\boldsymbol{M}$.

§. 4. An application of induced powers.

In this section, using the results of section 3, we shall prove the model existence theorem of $\boldsymbol{L}_{\omega_1,\omega}$. Let $\boldsymbol{\Gamma}$ be a set of formulas in $\boldsymbol{L}_{\omega_1,\omega}$ and ψ be a formula in $\boldsymbol{L}_{\omega_1,\omega}$. "$\psi$ is provable from $\boldsymbol{\Gamma}$", denoted by $\boldsymbol{\Gamma}\vdash\psi$, is defined as usual. Let S be a subset of $\boldsymbol{B}$. S is pairwise disjoint if $x\cdot y=0$ for any x, $y(\neq)$ in S. $\boldsymbol{B}$ satisfies the α-chain condition if every subset S of $\boldsymbol{B}$ which is pairwise disjoint has cardinality at most α.

THEOREM 4. *Let $\boldsymbol{\Gamma}$ be a set of sentences in $\boldsymbol{L}_{\omega_1,\omega}$. $\alpha=\mathrm{Max}\,(\mathrm{Card}\,(\boldsymbol{\Gamma}), \omega)$. Then, $\boldsymbol{\Gamma}$ is consistent iff there exists a complete Boolean algebra B satisfying the α-chain condition and a $\boldsymbol{B}$-valued structure $\boldsymbol{N}$ which is rigid and has a witness such that $|\psi|(\boldsymbol{N})=1$ for any ψ in $\boldsymbol{\Gamma}$.*

To prove Theorem 4, we need some preparations. Let ψ be in $\boldsymbol{L}_{\omega_1,\omega}$. We shall define $\mathrm{sub}\,(\psi)$ inductively as follows:

(1) $\mathrm{sub}\,(Pt_1\cdots t_k)=\{Pt_1\cdots t_k\}$,

(2) $\mathrm{sub}\,(\neg\psi)=\{\neg\psi\}\cup\mathrm{sub}\,(\psi)$,

(3) $\mathrm{sub}\,(\bigvee\psi)=\{\bigvee\psi\}\cup\bigcup\{\mathrm{sub}\,(\theta);\, \theta\in\psi\}$, and

(4) $\mathrm{sub}\,((\exists v)\psi)=\{(\exists v)\psi\}\cup\mathrm{sub}\,(\psi)$.

$\boldsymbol{\Gamma}'$ is the union of $\{\mathrm{sub}\,(\psi);\, \psi\in\boldsymbol{\Gamma}\}$. Choose A which is a countable set of new constant symbols. $\boldsymbol{\Gamma}_0$ is the set of sentences generated by the finite first order formulation from $\boldsymbol{\Gamma}'$ and A.

Define the relation $\cong$ on $\boldsymbol{\Gamma}_0$ as follows:

$$\psi\cong\theta \quad \text{iff} \quad \boldsymbol{\Gamma}\vdash\psi\leftrightarrow\theta$$

Cleary, $\cong$ is an equivalence relation on $\boldsymbol{\Gamma}_0$. Let $\psi/\!\cong=\{\theta\in\boldsymbol{\Gamma}_0;\, \psi\cong\theta\}$, and $\boldsymbol{B}_0=\boldsymbol{\Gamma}_0/\!\cong=\{\psi/\!\cong;\, \psi\in\boldsymbol{\Gamma}_0\}$. $\boldsymbol{B}_0$ becomes a Boolean algebra whose operations are defined by $(\psi/\!\cong)+(\theta/\!\cong)=(\psi\vee\theta/\!\cong)$,

$$(\psi/\!\cong)\cdot(\theta/\!\cong)=(\psi\wedge\theta/\!\cong) \quad \text{and} \quad -(\psi/\!\cong)=(\neg\psi/\!\cong) .$$

Let $\boldsymbol{B}$ be the canonical completion of $\boldsymbol{B}_0$. Since the cardinality of $\boldsymbol{B}_0$ is at most α, $\boldsymbol{B}$ satisfies the α-chain condition.

We define a $\boldsymbol{B}$-valued structure $\boldsymbol{M}$ for $\boldsymbol{L}(A)$ as follows:
(1) $M=\{t; t$ is a closed term in $\boldsymbol{L}(A).\}$,
(2) $\boldsymbol{M}(P)(t_1, \cdots, t_k)=Pt_1\cdots t_k/\cong$ for any P in $\boldsymbol{L}$,
(3) $\boldsymbol{M}(f)(t_1, \cdots, t_k)=ft_1\cdots t_k$ for any f in $\boldsymbol{L}$, and
(4) $\boldsymbol{M}(e)=e$ for any e in $\boldsymbol{L}(A)$.
Then, $[t](\boldsymbol{M})=t$ for any closed term t in $\boldsymbol{L}(A)(\boldsymbol{M})$.

LEMMA 6. *$|\psi|=\psi/\cong$ for any sentence ψ in $\boldsymbol{\Gamma}_0$.*

Proof. By induction on the construction of sentences in $\boldsymbol{\Gamma}_0$.
(1) ψ is atomic. The result follows from the definition.
(2) ψ is $\neg\theta$. By the induction hypothesis, $|\theta|=\theta/\cong$. So, $|\psi|=-|\theta|=-(\theta/\cong)=(\neg\theta/\cong)$.
(3) ψ is $(\exists v)\theta(v)$. By the induction hypothesis, $|\theta(t)|=\theta(t)/\cong$ for any t in M. So,

$$\begin{aligned}|\psi| &= |(\exists v)\theta(v)|\\ &= \sum\{|\theta(t)|;\ t\in M\}\\ &= \sum\{\theta(t)/\cong;\ t\in M\}\,.\end{aligned}$$

It is sufficient to show $\psi/\cong=\sum\{\theta((t)/\cong;\ t\in M\}$. Since $\vdash\theta(t)\longrightarrow\psi$ holds for any t in $\boldsymbol{L}(A)$, it follows that $\theta(t)/\cong\leqq\psi/\cong$. Choose a in A which is not contained in Γ, ψ and $\theta(v)$. $\boldsymbol{\Gamma}\vdash\theta(a)\longrightarrow\psi$ holds. So, $\boldsymbol{\Gamma}\vdash(\exists v)\theta(v)\longrightarrow\psi$ holds.
(4) ψ is $\vee\theta$. This case is similar to case (3).

Proof of Theorem 4. We take $\boldsymbol{B}$, $\boldsymbol{M}$ as above. Since every equality axiom is provable, and every sentence in $\boldsymbol{\Gamma}$ is provable from $\boldsymbol{\Gamma}$, from Lemma 6, it readily follows that $\boldsymbol{M}$ satisfies the equality exioms, and $|\psi|(\boldsymbol{M})=1$ for any ψ in $\boldsymbol{\Gamma}$. From Corollary 2, there exists a rigid witness structure X which in a $\boldsymbol{L}(A)_{\omega_1,\omega}$-elementary extension of $\boldsymbol{M}$. N is the restriction of X to $\boldsymbol{L}$. N is the desired one.

COROLLARY 3 (Model Existence Theorem). *Suppose that the cardinal of $\boldsymbol{\Gamma}$ is at most countable. If $\boldsymbol{\Gamma}$ is consistent, then $\boldsymbol{\Gamma}$ has a two-valued model.*

This Corollary is clear from theorem 3 and the Rasiowa-Sikorski's Lemma

Reference

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